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Reduced-Order Multirate Estimation

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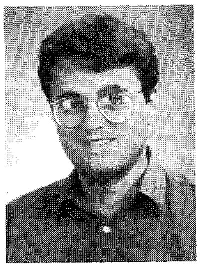
In this paper we develop an approach to designing reduced-order multirate estimators. A discrete-time model that accounts for the multirate timing sequence of measurements is presented and is shown to have periodically time-varying dynamics. Using discrete-time stability theory, the optimal projection approach to fixed-order (i.e., full- and reduced-order) estimation is generalized to obtain reduced-order periodic estimators that account for the multirate architecture. It is shown that the optimal reduced-order filter is characterized by means of a periodically time-varying system of equations consisting of coupled Riccati and Lyapunov equations. A novel homotopy algorithm, based on a Newton correction scheme, is also presented which allows solutions to periodic difference Riccati equations.



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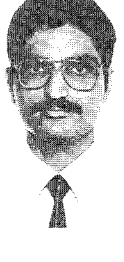

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Nomenclature

A, $C(k)$	= $n \times n$, $l_k \times n$ matrices
$A_e(k)$, $B_e(k)$, $C_e(k)$	= $n_e \times n_e$, $n_e \times l_k$, $q \times n_e$ matrices
$\mathscr{E}$	= expected value
I_r, $0_{r\times s}$, 0_r	= $r \times r$ identity matrix, $r \times s$ zero matrix, $0_{r\times r}$
k, α	= discrete-time indices
L	= $q \times n$ matrix
n, l_k, n_e	= positive integers, $1 \le n_e \le n$
$\tilde{n}$	= $n + n_e$
R	= $q \times q$ positive-definite matrix
$\mathscr{R}$, $\mathscr{R}^{r\times s}$, $\mathscr{R}^r$	= real numbers, $r \times s$ real matrices, $\mathscr{R}^{r\times 1}$
x, y, x_e, y_e	= n-, l_k-, n_e-, q-dimensional vectors
$\rho()$	= spectral radius
$()^T$, $()^{-1}$	= transpose, inverse

I. Introduction

IN practical applications, aerospace systems often involve sensors operating at different sampling rates. To properly use such data, a multirate filter or state estimator must carefully account for the timing sequence of the incoming data. The purpose of this paper is to develop a general approach to full- and reduced-order steady-state multirate estimation.

Despite the widespread usage of Kalman filters and state estimators, the research literature on multirate filtering and estimation is rather limited. Notable exceptions include Refs. 1–4 which address the multirate problem. A common feature of these papers as well as Refs. 5–7 is the realization that the multirate sampling process leads to a periodically time-varying discrete-time dynamic model. Hence with suitable reinterpretation, results on multirate estimation can also be applied to single rate or multirate problems involving systems with periodically time-varying dynamics. Such connections will be more fully explored in a future paper. Similarly, extensions to the multirate feedback control problem are also outside the scope of this paper. The interested reader is referred to Refs. 8–12 for further discussions on periodic and multirate control.

For generality in our development, we consider both full- and reduced-order filters. In the discrete-time case this problem was considered in Ref. 13 whereas sampled-data aspects were addressed in Ref. 14. Reference 15 discusses the motivation for implementing reduced-order estimators and, along with Ref. 16, contains an extensive bibliography relating to the reduced-order problem. In the full-order, strictly proper estimator case our results essentially correspond to Refs. 2 and 3.

The approach of the present paper is the Riccati equation technique developed in Ref. 16. There it was shown that optimal reduced-order, steady-state estimators can be characterized by means of an algebraic system of equations consisting of one modified Riccati equation and two modified Lyapunov equations coupled by a projection matrix τ. The coupling via the projection τ illustrates the fact that three matrix equations characterize the optimal reduced-order state estimator with intrinsic coupling between the operations of optimal estimator design and optimal estimator reduction.

In Ref. 13 the discrete-time problem was addressed without any reference to sampled-data design. Furthermore, the discrete-time model was assumed to be time-invariant and steady-state reduced-order estimators were sought. In the present paper, however, the analog-to-digital (A/D) conversions are employed within a multirate setting to obtain periodically time-varying dynamics. The estimator is thus assigned a corresponding discrete-time periodic structure to account for the multirate measurements. It is shown that the optimal reduced-order multirate estimator is characterized by a periodically time-varying system of three equations consisting of one modified Riccati equation and two modified Lyapunov equations corresponding to each intermediate point of the periodicity interval. Because of the time-varying nature of the problem, the necessary conditions for optimality now involve projections corresponding to intermediate points of the periodic interval.

The contents of this paper are as follows. In Sec. II, the statement of the reduced-order multirate estimation problem is presented. Lemma 1, which gives the central result of Sec. II, shows that under the assumption of cyclostationary disturbances the error covariance equation reaches a steady-state periodic trajectory under periodic dynamics. In Sec. III, Theorem 2 presents necessary conditions for optimality which characterize solutions to the multirate estimation problem. In Sec. IV, using the identities of Van Loan,[17] we derive formulas for integrals of matrix exponentials arising in the continuous-time/sampled-data conversion. To illustrate these results we describe a numerical algorithm in Sec. V for solving the design equations and apply the algorithm to an illustrative numerical example. Finally, Sec. VI gives some conclusions and discusses future extensions.

II. Reduced-Order Multirate Estimation Problem

In this section we state the fixed-order, sampled-data, multirate estimation problem. In the problem formulation the sample intervals h_k and the estimator order n_e are fixed, and the optimization is performed over the estimator parameters $[A_e(\cdot), B_e(\cdot), C_e(\cdot)]$. For design tradeoff studies h_k and n_e can be varied, and the problem can be solved for each pair of values of interest. Finally, we assume that the plant dynamics A is asymptotically stable. The case in which A may contain unstable modes (e.g., rigid body dynamics) is significantly more involved and is deferred to a future paper. For details on the unstable reduced-order estimation problem see Ref. 16.

Consider the nth-order system

$$\dot{x}(t) = Ax(t) + w_1(t), \qquad t \in [0_1, \infty) \tag{1}$$

where A is stable, with multirate sampled-data measurements

$$y(t_k) = C(t_k)x(t_k) + w_2(t_k), \qquad k = 1, 2, \ldots \tag{2}$$

Then design an n_eth-order ($1 \le n_e \le n$) sampled-data estimator

$$x_e(t_k + 1) = A_e(t_k)x_e(t_k) + B_e(t_k)y(t_k), \quad 0 < t_1 < t_2 < \cdots \tag{3}$$

$$y_e(t_k) = C_e(t_k)x_e(t_k) \tag{4}$$

$$y_e(t) = y_e(t_k), \qquad t \in [t_k, t_{k+1}) \tag{5}$$

that minimizes the least-squares, steady-state estimation-error criterion

$$J_e[A_e(\cdot), B_e(\cdot), C_e(\cdot)] \triangleq \lim_{t\to\infty} \mathscr{E}\frac{1}{t}\int_0^t [\{Lx(s) - y_e(s)\}^T R\{Lx(s) - y_e(s)\}]\,ds \tag{6}$$

In Eq. (6) the matrix L identifies the linear combinations Lx of states x whose estimates are desired. The key feature of this problem is the time-varying nature of the output equation (2) that represents sensor measurements available at different rates. Figure 1 provides a typical multirate timing diagram for a three-sensor model. For generality, we do not assume that the sample intervals $h_k \triangleq t_{k+1} - t_k$ are uniform (note the sample times for sensor 3 in Fig. 1). However, we do assume that the overall timing sequence of intervals $[t_k, t_{k+N}]$, $k = 1, 2, \ldots$, is periodic over $[0, \infty)$, where N represents the periodic interval. Note that $h_{k+N} = h_k$, $k = 1, 2, \ldots$. Since different sensor measurements are available at different times t_k, the dimension l_k of the measurements $y(t_k)$ may also vary periodically. Finally, in subsequent analysis the estimator (3) and (4) is assigned periodic gains corresponding to the periodic timing sequence of the multirate measurements.

In the problem formulation, $w_1(t)$ denotes a continuous-time, white noise process with non-negative-definite intensity $V_1 \in$

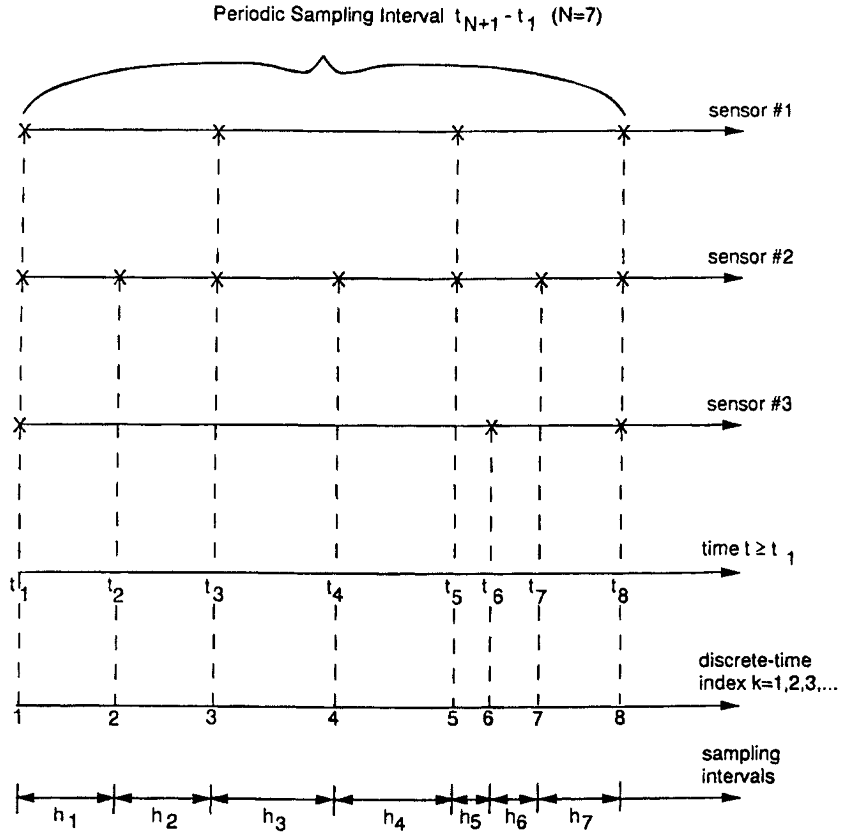

Fig. 1 Multirate timing diagram for sampled-data estimation.

$\mathcal{R}^{n\times n}$, whereas $w_2(t_k)$ denotes a variable-dimension, discrete-time white noise process with positive-definite covariance $V_2(t_k) \in \mathcal{R}^{l_k \times l_k}$. We assume $w_2(t_k)$ is cyclostationary, that is, $V_2(t_{k+N}) = V_2(t_k)$, $k = 1, 2, \ldots$.

In what follows we shall simplify the notation considerably by replacing the real-time sample instant t_k by the discrete-time index k. With this minor abuse of notation we replace $x(t_k)$ by $x(k)$, $x_e(t_k)$ by $x_e(k)$, $y(t_k)$ by $y(k)$, $w_2(t_k)$ by $w_2(k)$, $A_e(t_k)$ by $A_e(k)$ [and similarly for $B_e(\cdot)$ and $C_e(\cdot)$], $C(t_k)$ by $C(k)$, and $V_2(t_k)$ by $V_2(k)$. The context should clarify whether the argument denotes k or t_k. With this notation our periodicity assumption on the estimator implies $A_e(k+N) = A_e(k)$, $k = 1, 2 \ldots$, and similarly for $B_e(\cdot)$ and $C_e(\cdot)$. Also, by assumption, $C(k+N) = C(k)$, $k = 1, 2, \ldots$.

Next, we model the propagation of the plant over one time step. For notational convenience define

$$H(k) \triangleq \int_0^{h_k} e^{As}\, ds$$

Theorem 1. For the reduced-order multirate estimation problem, the plant dynamics (1) and the least-squares state-estimation criterion (6) have the equivalent discrete-time representations

$$x(k+1) = A(k)x(k) + w_1'(k) \tag{7}$$

$$y(k) = C(k)x(k) + w_2(k) \tag{8}$$

$$J_e[A_e(\cdot), B_e(\cdot), C_e(\cdot)] = \delta_\infty + \lim_{K\to\infty} \mathcal{E} \frac{1}{K}\sum_{k=1}^{K} \int_0^{h_k} [Le^{As}x(k) - y_e(k)]^T R(k) \times [Le^{As}x(k) - y_e(k)]\, ds \tag{9}$$

where

$$A(k) \triangleq e^{Ah_k}, \qquad \delta_\infty \triangleq \lim_{K\to\infty} \frac{1}{K} \operatorname{tr} \sum_{k=1}^{K} \frac{1}{h_k} \times \int_0^{h_k}\int_0^{s} e^{Ar}V_1 e^{A^T r} L^T R L\, dr\, ds, \quad R(k) \triangleq \frac{1}{h_k} R \tag{10}$$

and $w_1'(k)$ is a zero-mean, discrete-time white noise process with

$$\mathcal{E}\{w_1'(k) w_1'^T(k)\} = V_1(k)$$
$$V_1(k) \triangleq \int_0^{h_k} e^{As} V_1 e^{A^T s}\, ds \tag{11}$$

Note that by the sampling periodicity assumption, $A(k+N) = A(k)$, $k = 1, 2 \ldots$. The proof of this theorem is a straightforward calculation involving integrals of white noise signals and hence is omitted. See Ref. 14 for related details.

The above formulation assumes that a discrete-time multirate measurement model is available. One can assume, alternatively, that analog measurements corrupted by continuous-time white noise are available instead, that is, $y(t) = Cx(t) + w_2(t)$. In this case one can develop an equivalent discrete-time model that employs an averaging-type A/D device[14,18-20]

$$\hat{y}(k) = \frac{1}{h_k}\int_{t_k}^{t_{k+1}} y(t)\, dt \tag{12}$$

It can be shown that the resulting averaged measurements depend on delayed samples of the state. In this case the equivalent discrete-time model can be captured by a suitably augmented system. For details see Refs. 14 and 19.

Remark 1. The equivalent discrete-time, least-squares estimation-error criterion (9) involves a constant offset δ_∞ which is a function of sample rates and effectively imposes a lower bound on the sampled-data performance due to the discretization process. (As will be shown by Lemma 1, due to the periodicity of h_k, δ_∞ is a constant.)

Next, we combine Eqs. (3), (4), (7), and (8) to form the augmented equivalent discrete-time periodic system

$$\tilde{x}(k+1) = \tilde{A}(k)\tilde{x}(k) + \tilde{w}(k) \tag{13}$$

where

$$\tilde{x}(k) \triangleq \begin{bmatrix} x(k) \\ x_e(k) \end{bmatrix}, \quad \tilde{A}(k) \triangleq \begin{bmatrix} A(k) & 0 \\ B_e(k)C(k) & A_e(k) \end{bmatrix}$$
$$\tilde{A}(k+N) = \tilde{A}(k), \qquad k = 1, 2, \ldots \tag{14}$$

The augmented disturbance

$$\tilde{w}(k) \triangleq \begin{bmatrix} w_1'(k) \\ B_e(k)w_2(k) \end{bmatrix} \tag{15}$$

has non-negative-definite covariance

$$\tilde{V}(k) \triangleq \begin{bmatrix} V_1(k) & V_{12}(k)B_e^T(k) \\ B_e(k)V_{12}^T(k) & B_e(k)V_2(k)B_e^T(k) \end{bmatrix} \tag{16}$$

where $V_{12}(k) \triangleq \mathscr{E}[w_1'(k)w_2^T(k)]$ denotes the noise correlation between the plant disturbance and measurement noise.

The state estimation error criterion (9) can now be expressed in terms of the augmented second-moment matrix. The following result is immediate.

Proposition 1. For given $[A_e(\cdot), B_e(\cdot), C_e(\cdot)]$ the second-moment matrix

$$\tilde{Q}(k) \triangleq \mathscr{E}[\tilde{x}(k)\tilde{x}^T(k)] \tag{17}$$

satisfies

$$\tilde{Q}(k+1) = \tilde{A}(k)\tilde{Q}(k)\tilde{A}^T(k) + \tilde{V}(k) \tag{18}$$

Furthermore,

$$J_e[A_e(\cdot), B_e(\cdot), C_e(\cdot)] = \delta_\infty + \lim_{K\to\infty} \frac{1}{K} \operatorname{tr} \sum_{k=1}^{K} [\tilde{Q}(k)\tilde{R}(k)] \tag{19}$$

where

$$\tilde{R}(k) \triangleq \begin{bmatrix} R_1(k) & R_{12}(k) \\ R_{12}^T(k) & R_2(k) \end{bmatrix} \tag{20}$$

and

$$R_1(k) \triangleq \int_0^{h_k} e^{A^Ts}L^TR(k)Le^{As}\, ds$$

$$R_{12}(k) \triangleq -H^T(k)L^TR(k)C_e(k)$$

$$R_2(k) \triangleq C_e^T(k)RC_e(k)$$

Remark 2. Note that Eq. (18) is a periodic Lyapunov equation. Relevant references include Refs. 21–25.

We now show that the covariance Lyapunov equation (18) reaches a steady-state periodic trajectory as $K \to \infty$. For the next result we introduce the parametrization, $k = \alpha + \beta N$, where the index α satisfies $1 \le \alpha \le N$, and $\beta = 1, 2, \ldots$. We now restrict our attention to estimators having the property that the estimator transition matrix over one period

$$\Phi_{ep}(\alpha) \triangleq A_e(\alpha + N - 1)A_e(\alpha + N - 2) \cdots A_e(\alpha) \tag{21}$$

is stable for $\alpha = 1, \ldots, N$. Note that since $A_e(\cdot)$ is required to be periodic, the eigenvalues of $\Phi_{ep}(\alpha)$ are actually independent of α. Hence, it suffices to require that $\Phi_{ep}(1) = A_e(N)A_e(N-1) \cdots A_e(1)$ is stable. Next, defining the plant transition matrix over one period

$$\Phi_p(\alpha) \triangleq A(\alpha + N - 1)A(\alpha + N - 2) \cdots A(\alpha) \tag{22}$$

we see that

$$\Phi_p(\alpha) = e^{A(t_{N+1} - t_1)}, \qquad \alpha = 1, \ldots, N \tag{23}$$

Since A is assumed to be (continuous-time) stable, it follows that $\Phi_p(\alpha)$ is (discrete-time) stable. Finally, we define the transition matrix over one period for the augmented system (13) by

$$\tilde{\Phi}_p(\alpha) \triangleq \tilde{A}(\alpha + N - 1)\tilde{A}(\alpha + N - 2) \cdots \tilde{A}(\alpha) \tag{24}$$

Since $\tilde{A}(\cdot)$ is lower block triangular and $\Phi_p(\cdot)$ and $\Phi_{ep}(\cdot)$ are stable, it follows from the structure of $\tilde{\Phi}_p(\alpha)$ that $\tilde{\Phi}_p(\alpha)$ is also stable for $\alpha = 1, \ldots, N$.

Lemma 1. For given $[A_e(k), B_e(k), C_e(k)]$ the covariance Lyapunov equation (18) reaches a steady-state periodic trajectory as $k \to \infty$, that is,

$$\lim_{k\to\infty} [\tilde{Q}(k), \tilde{Q}(k+1), \ldots, \tilde{Q}(k+N-1)]$$
$$= [\tilde{Q}(\alpha), \tilde{Q}(\alpha+1), \ldots, \tilde{Q}(\alpha+N-1)] \tag{25}$$

In this case the covariance $\tilde{Q}(k)$ defined by Eq. (17) satisfies

$$\tilde{Q}(\alpha+1) = \tilde{A}(\alpha)\tilde{Q}(\alpha)\tilde{A}^T(\alpha) + \tilde{V}(\alpha), \qquad \alpha = 1, \ldots, N \tag{26}$$

where

$$\tilde{Q}(N+1) = \tilde{Q}(1) \tag{27}$$

Furthermore, the quadratic least-squares error criterion (19) is given by

$$J_e[A_e(\cdot), B_e(\cdot), C_e(\cdot)] = \delta + \frac{1}{N} \operatorname{tr} \sum_{\alpha=1}^{N} [\tilde{Q}(\alpha)\tilde{R}(\alpha)] \tag{28}$$

where

$$\delta \triangleq \frac{1}{N} \operatorname{tr} \sum_{\alpha=1}^{N} \frac{1}{h_\alpha} \int_0^{h_\alpha} \int_0^{s} e^{Ar}V_1e^{A^Tr}L^TRL\, dr\, ds \tag{29}$$

Proof: The proof is given in Appendix A. □

III. Necessary Conditions for the Reduced-Order Multirate Estimation Problem

In this section we obtain necessary conditions that characterize solutions to the reduced-order multirate estimation problem. Derivation of these conditions requires additional technical assumptions. Specifically, we further restrict $[A_e(\cdot), B_e(\cdot), C_e(\cdot)]$ to the set

$$\mathfrak{S} \triangleq \{[A_e(\cdot), B_e(\cdot), C_e(\cdot)] : \tilde{\Phi}_p(\alpha) \text{ is stable and}$$
$$[\Phi_{ep}(\alpha), B_{ep}(\alpha), C_{ep}(\alpha)] \text{ is minimal}, \alpha = 1, \ldots, N\} \tag{30}$$

where

$$B_{ep}(\alpha) \triangleq [A_e(\alpha+N-1)A_e(\alpha+N-2) \cdots A_e(\alpha+1)B_e(\alpha), A_e(\alpha+N-1)A_e(\alpha+N-2) \cdots A_e(\alpha+2)B_e(\alpha+1), \ldots, B_e(\alpha+N-1)] \quad (31)$$

$$C_{ep}(\alpha) \triangleq \begin{bmatrix} C_e(\alpha+N-1)A_e(\alpha+N-2)\cdots A_e(\alpha) \\ C_e(\alpha+N-2)A_e(\alpha+N-3)\cdots A_e(\alpha) \\ \vdots \\ C_e(\alpha) \end{bmatrix} \quad (32)$$

As can be seen from Appendix B, the set $\mathfrak{S}$ constitutes sufficient conditions under which the Lagrange multiplier technique is applicable to the reduced-order multirate estimation problem. This is similar to concepts involving the moving equilibrium for periodic Lyapunov/Riccati equations discussed in Refs. 23 and 24. Specifically, the formulas for the lifted isomorphism (31) and (32) are equivalent to assuming the stability of $\tilde{A}(\cdot)$ along with the reachability and observability of $[A_e(\cdot), B_e(\cdot), C_e(\cdot)]$ (Refs. 10 and 23). The asymptotic stability of the transition matrix $\tilde{\Phi}_p(\alpha)$ serves as a normality condition which further implies that the dual $\tilde{P}(\alpha)$ of $\tilde{Q}(\alpha)$ is non-negative-definite. Furthermore, the assumption that $[\Phi_{ep}(\alpha), B_{ep}(\alpha), C_{ep}(\alpha)]$ is controllable and observable is a nondegeneracy condition which implies that the lower right $n_e \times n_e$ subblocks of $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha)$ are positive definite thus yielding explicit gain expressions for $A_e(\alpha)$, $B_e(\alpha)$, and $C_e(\alpha)$.

To state the main results we require some additional notation and a lemma concerning pairs of non-negative-definite matrices. See Ref. 26 for details.

Lemma 2. Let $\hat{Q}, \hat{P}$ be $n \times n$ non-negative-definite matrices and assume rank $\hat{Q}\hat{P} = n_e$. Then there exist $n_e \times n$ matrices G, Γ and an $n_e \times n_e$ invertible matrix M, unique except for a change of basis in $\mathfrak{R}^{n_e}$, such that

$$\hat{Q}\hat{P} = G^T M \Gamma \quad (33a)$$

$$\Gamma G^T = I_{n_e} \quad (33b)$$

Furthermore, the $n \times n$ matrices

$$\tau \triangleq G^T\Gamma, \qquad \tau_\perp \triangleq I_n - \tau \quad (34)$$

are idempotent and have rank n_e and $n - n_e$, respectively.

The following results give necessary conditions that characterize solutions to the reduced-order multirate estimation problem. For convenience in stating this result, define the notation

$$V_{2a}(\alpha) \triangleq C(\alpha)Q(\alpha)C^T(\alpha) + V_2(\alpha)$$

$$Q_a(\alpha) \triangleq A(\alpha)\,Q(\alpha)C^T(\alpha) + V_{12}(\alpha) \quad (35)$$

$$\hat{L}(\alpha) \triangleq (1/h_\alpha)LH(\alpha)$$

for arbitrary $Q(\alpha) \in \mathfrak{R}^{n\times n}$ and $\alpha = 1, \ldots, N$.

Theorem 2. Suppose $[A_e(\cdot), B_e(\cdot), C_e(\cdot)] \in \mathfrak{S}$ solves the reduced-order multirate estimation problem. Then there exist $n \times n$ non-negative-definite matrices $Q(\alpha)$, $\hat{Q}(\alpha)$, and $\hat{P}(\alpha)$ such that, for $\alpha = 1, \ldots, N$, $A_e(\alpha)$, $B_e(\alpha)$ and $C_e(\alpha)$ are given by

$$A_e(\alpha) = \Gamma(\alpha+1)[A(\alpha) - Q_a(\alpha)V_{2a}^{-1}(\alpha)C(\alpha)]G^T(\alpha) \quad (36)$$

$$B_e(\alpha) = \Gamma(\alpha+1)Q_a(\alpha)V_{2a}^{-1}(\alpha) \quad (37)$$

$$C_e(\alpha) = \hat{L}(\alpha)G^T(\alpha) \quad (38)$$

and such that $Q(\alpha)$, $\hat{Q}(\alpha)$, and $\hat{P}(\alpha)$ satisfy

$$Q(\alpha+1) = A(\alpha)Q(\alpha)A^T(\alpha) + V_1(\alpha) - Q_a(\alpha)V_{2a}^{-1}(\alpha)Q_a^T(\alpha) + \tau_\perp(\alpha+1) \times [A(\alpha)\hat{Q}(\alpha)A^T(\alpha) + Q_a(\alpha)V_{2a}^{-1}(\alpha)Q_a^T(\alpha)] \times \tau_\perp^T(\alpha+1) \quad (39)$$

$$\hat{Q}(\alpha+1) = \tau(\alpha+1)[A(\alpha)\hat{Q}(\alpha)A^T(\alpha) + Q_a(\alpha)V_{2a}^{-1}(\alpha)Q_a^T(\alpha)]\tau^T(\alpha+1) \quad (40)$$

$$\hat{P}(\alpha) = \tau^T(\alpha)[\{A(\alpha) - Q_a(\alpha)V_{2a}^{-1}(\alpha)C(\alpha)\}^T \times \hat{P}(\alpha+1)\{A(\alpha) - Q_a(\alpha)V_{2a}^{-1}(\alpha)C(\alpha)\} + (1/N)\{\hat{L}^T(\alpha)R\hat{L}(\alpha)\}]\tau(\alpha) \quad (41)$$

$$\operatorname{rank}\hat{Q}(\alpha) = \operatorname{rank}\hat{P}(\alpha) = \operatorname{rank}\hat{Q}(\alpha)\hat{P}(\alpha) = n_e \quad (42)$$

Furthermore, the minimal cost is given by

$$J_e[A_e(\cdot), B_e(\cdot), C_e(\cdot)] = \delta + \frac{1}{N}\operatorname{tr}\sum_{\alpha=1}^{N}[\{Q(\alpha) + \hat{Q}(\alpha)\}R_1(\alpha) - \hat{Q}(\alpha)\hat{L}^T(\alpha)R\hat{L}(\alpha)] \quad (43)$$

Proof: The proof is given in Appendix B. □

Theorem 2 presents necessary conditions for the reduced-order multirate estimation problem. These necessary conditions consist of a system of one modified Riccati equation and two modified Lyapunov equations coupled by projection matrices $\tau(\alpha)$, $\alpha = 1, \ldots, N$. As expected, these equations are periodically time varying over the period $1 \le \alpha \le N$ in accordance with the multirate nature of the measurements. As discussed in Ref. 15 the fixed-order constraint on the estimator gives rise to the projection τ which characterizes the optimal reduced-order estimator gains. In the multirate case, however, it is interesting to note that the time-varying nature of the solution involves multiple projections corresponding to each of the intermediate points of the periodicity interval.

Remark 3. As in the linear time-invariant case[13,15] to obtain the full-order multirate Kalman filter result, set $n_e = n$. In this case, $\tau(\alpha) = G(\alpha) = \Gamma(\alpha+1) = I_n$ for all $\alpha = 1, \ldots, N$. Furthermore, in this case Eqs. (40) and (41) are superfluous and can be omitted. Thus, as expected the optimal full-order multirate estimator is characterized by means of a single time-periodic Riccati equation (observer Riccati equation) over the period $\alpha = 1, \ldots, N$. Specifically, for $L = I_n$, the multirate Kalman filter is characterized by

$$A_e(\alpha) = A(\alpha) - Q_a(\alpha)V_{2a}^{-1}(\alpha)C(\alpha) \quad (44)$$

$$B_e(\alpha) = Q_a(\alpha)V_{2a}^{-1}(\alpha) \quad (45)$$

$$C_e(\alpha) = \hat{L}(\alpha) \quad (46)$$

where $Q(\alpha)$ satisfies

$$Q(\alpha+1) = A(\alpha)Q(\alpha)A^T(\alpha) + V_1(\alpha) - Q_a(\alpha)V_{2a}^{-1}(\alpha)Q_a^T(\alpha) \quad (47)$$

Note that if the plant model is assumed to be time-invariant, Eq. (47) collapses to the standard observer Riccati equation. Alternatively, if we retain the reduced-order constraint and assume time-invariant plant dynamics, Theorem 2 yields the linear time-invariant, discrete-time optimal projection equations for reduced-order estimation.[13,14]

IV. Numerical Evaluation of Integrals Involving Matrix Exponentials

To evaluate the integrals involving matrix exponentials appearing in Theorem 1, we utilize the approach of Ref. 17.

The idea is to eliminate the need for integration by computing the matrix exponential of appropriate block matrices. Numerical matrix exponentiation is discussed in Ref. 27.

Proposition 2. Consider the following partitioned matrix exponentials of orders $3n \times 3n$, $2n \times 2n$, and $(n + q) \times (n + q)$, respectively

$$\begin{bmatrix} F_1 & F_2 & F_3 \\ 0_n & F_4 & F_5 \\ 0_n & 0_n & F_6 \end{bmatrix} \triangleq \exp\begin{bmatrix} -A & I_n & 0_n \\ 0_n & -A & V_1 \\ 0_n & 0_n & A^T \end{bmatrix} h_\alpha$$

$$\begin{bmatrix} F_7 & F_8 \\ 0_n & F_9 \end{bmatrix} \triangleq \exp\begin{bmatrix} -A^T & L^T R L \\ 0_n & A \end{bmatrix} h_\alpha$$

$$\begin{bmatrix} F_{10} & F_{11} \\ 0_{q\times n} & I_q \end{bmatrix} \triangleq \exp\begin{bmatrix} A^T & L^T \\ 0_{q\times n} & 0_q \end{bmatrix} h_\alpha$$

for $\alpha = 1, \ldots, N$. Then

$$A(\alpha) = F_6^T, \qquad \hat{L}(\alpha) = (1/h_\alpha)F_{11}^T$$

$$R_1(\alpha) = (1/h_\alpha)F_9^T F_8, \qquad V_1(\alpha) = F_6^T F_5$$

$$\delta = \frac{1}{N}\sum_{\alpha=1}^{N} \frac{1}{h_\alpha} \operatorname{tr} L^T R L F_6^T F_3$$

for $\alpha = 1, \ldots, N$.

The proof of this proposition involves straightforward manipulations of matrix exponentials and hence is omitted.

V. Numerical Algorithm and Illustrative Numerical Example

In this section we present a numerical algorithm using homotopic continuation techniques[28] for solving the design equation (47) for the full-order multirate estimation problem and consider an illustrative numerical example. In the following we use the notation $Q_\alpha \triangleq Q(\alpha)$.

To solve Eq. (47) for $\alpha = 1, \ldots, N$, consider the equivalent discrete-time algebraic Riccati equation[24]

$$Q_\alpha = \bar{\Phi}_{\alpha+N,\alpha} Q_\alpha \bar{\Phi}^T_{\alpha+N,\alpha} + W_{\alpha+N,\alpha} \tag{48}$$

where

$$\bar{\Phi}_{\alpha+N,i} \triangleq A(\alpha+N-i)A(\alpha+N-(i+1))\cdots A(i)$$
$$\alpha + N > i, \qquad \bar{\Phi}_{\alpha+N,\alpha+N} = I_{n^2} \tag{49}$$

and $W_{\alpha+N,\alpha}$ is the reachability Gramian defined by

$$W_{\alpha+N,\alpha} \triangleq \sum_{i=\alpha}^{\alpha+N-1} \{\bar{\Phi}_{\alpha+N,i+1}[V_{1_i} - Q_{a_i} V_{2a_i}^{-1} Q_{a_i}^T]\bar{\Phi}^T_{\alpha+N,i+1}\} \tag{50}$$

Next we form the homotopy map for Eq. (48) as follows:

$$(1-\beta)E_Q = \bar{\Phi}_{\alpha+N,\alpha} Q_\alpha(\beta) \bar{\Phi}^T_{\alpha+N,\alpha} + W_{\alpha+N,\alpha}(\beta) - Q_\alpha(\beta) \tag{51}$$

where E_Q is the error in Eq. (48) with current approximation for Q_α for $\alpha = 1, \ldots, N$; $\beta \in [0, 1]$ is the homotopy parameter; and

$$W_{\alpha+N,\alpha}(\beta) \triangleq \sum_{i=\alpha}^{\alpha+N-1} \{\bar{\Phi}_{\alpha+N,i+1}[V_{1_i} - Q_{a_i}(\beta) V_{2a_i}^{-1}(\beta) Q_{a_i}^T(\beta)]\bar{\Phi}^T_{\alpha+N,i+1}\}$$

where

$$V_{2a_i}(\beta) \triangleq C_i Q_i(\beta) C_i^T + V_{2_i}$$

$$Q_{a_i}(\beta) \triangleq A_i Q_i(\beta) C_i^T + V_{12_i}$$

Differentiating Eq. (51) with respect to β and using the approach of Ref. 28 gives the Newton correction equation

$$\Delta Q(\alpha) = \bar{\mathcal{A}} \Delta Q(\alpha) \bar{\mathcal{A}}^T + E_Q \tag{52}$$

where

$$\bar{\mathcal{A}} \triangleq \bar{\Phi}_{\alpha+N,\alpha+1}[A(\alpha) - Q_a(\alpha) V_{2a}^{-1}(\alpha) C(\alpha)]$$

Note that Eq. (52) is a discrete-time algebraic Lyapunov equation.

Algorithm 1. To solve the design equation (47), carry out the following steps.

1) Initialize $Q(\alpha)$ for $\alpha = 1, \ldots, N$.
2) Compute the error E_Q in Eq. (48). If E_Q satisfies the final tolerance then stop.
3) Solve Eq. (52) to obtain a Newton homotopy correction.
4) Let $Q(\alpha) \leftarrow Q(\alpha) + \Delta Q(\alpha)$.
5) Propagate Eq. (47) over the period.
6) Compute the error E_Q in Eq. (48). If E_Q satisfies some preassigned tolerance then go to step 7 else reduce $\Delta Q(\alpha)$ and go to step 4.
7) If E_Q satisfies the final tolerance then stop else go to step 3.

For illustrative purposes consider a simply supported Euler-Bernoulli beam. The partial differential equation for the transverse deflection $w(x, t)$ is given by

$$m(x)\frac{\partial^2 w(x,t)}{\partial t^2} = -\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] + f(x,t)$$
$$w(x,t)|_{x=0,L} = 0, \qquad EI(x)\left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0,L} = 0 \tag{53}$$

where $m(x)$ is the mass per unit length of the beam, $EI(x)$ is the flexural rigidity with E denoting Young's modulus of elasticity and $I(x)$ denoting the moment of inertia about an axis normal to the plane of vibration and passing through the center of the cross-sectional area. Finally, $f(x, t)$ is a distributed disturbance acting on the beam. Assuming uniform beam properties, the modal decomposition of this system has the form

$$w(x,t) = \sum_{r=1}^{\infty} W_r(x) q_r(t), \qquad \int_0^L m W_r^2(x)\,dx = 1$$

$$W_r(x) = \sqrt{\frac{2}{mL}} \sin\frac{r\pi x}{L}$$

where, assuming uniform proportional damping, the modal coordinates q_r satisfy

$$\ddot{q}_r(t) + 2\zeta\omega_r\dot{q}_r(t) + \omega_r^2 q_r(t) = \int_0^L f(x,t) W_r(x)\,dx$$
$$r = 1, 2, \ldots \tag{54}$$

For simplicity assume $L = \pi$ and $m = EI = 2/\pi$ so that $\sqrt{2/mL} = 1$. Furthermore, we assume two sensors located at $x = 0.55\pi$ and $x = 0.65\pi$ are sampling at 60 Hz and 30 Hz, respectively. Also,it is assumed that a white noise disturbance of unit intensity acts on the beam at $x = 0.45\pi$. As inputs to the estimator design we chose to weight the performance of the beam displacement at $x = 0.65\pi$. Finally, modeling the first five modes and defining the plant states as $x = [q_1, \dot{q}_1, \ldots, q_5, \dot{q}_5]^T$, the resulting state-space model is

$$A = \underset{i=1,\ldots,5}{\text{block-diag}}\begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta\omega_i \end{bmatrix}, \qquad \omega_i = i^2, \qquad i = 1,\ldots,5, \qquad \zeta = 0.005$$

$$C = \begin{bmatrix} 0.9877 & 0 & -0.3090 & 0 & -0.8910 & 0 & 0.5878 & 0 & 0.7071 & 0 \\ 0.8910 & 0 & -0.8090 & 0 & -0.1564 & 0 & 0.9511 & 0 & -0.7071 & 0 \end{bmatrix}$$

$$L = [0.8910 \quad 0 \quad -0.8090 \quad 0 \quad -0.1564 \quad 0 \quad 0.9511 \quad 0 \quad -0.7071 \quad 0]$$

$$D_1 = [0 \quad 0.9877 \quad 0 \quad 0.3090 \quad 0 \quad -0.8900 \quad 0 \quad -0.5878 \quad 0 \quad 0.7071]^T$$

$$V_2 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}, \qquad R = 0.1$$

For $n_e = 10$, discrete-time single rate and multirate estimators were obtained from Eqs. (44–47) using Theorem 1 for continuous-time to discrete-time conversions. Different measurement schemes were considered, and the resulting designs are compared in Figs. 2 and 3. The results are summarized as follows. Figures 2 and 3 show simulation plots for error states 7 and 8, in the presence of zero mean unit intensity white noise input disturbances, respectively. Finally, five designs were compared using the performance criterion (43). The results are summarized in Table 1.

It is interesting to note that the multirate architecture gives the least cost for the cases considered with the exception to the two 60-Hz sensor scheme which is to be expected. In this case, the improvement in the cost of two 60-Hz sensor scheme over the multirate scheme is minimal. However, the multirate scheme provides sensor complexity reduction over the two 60-Hz sensor scheme.

Table 1 Design schemes

Measurement scheme	Optimal cost
One 30-Hz sensor at $x = 0.65\pi$	1.000×10^{-3}
Two 30-Hz sensors at $x = 0.55\pi$ and $x = 0.65\pi$	6.3313×10^{-4}
One 60-Hz sensor at $x = 0.65\pi$	4.3110×10^{-4}
Two 60-Hz sensors at $x = 0.55\pi$ and $x = 0.65\pi$	2.1775×10^{-4}
Multirate scheme (30-Hz and 60-Hz sensors)	2.4286×10^{-4}

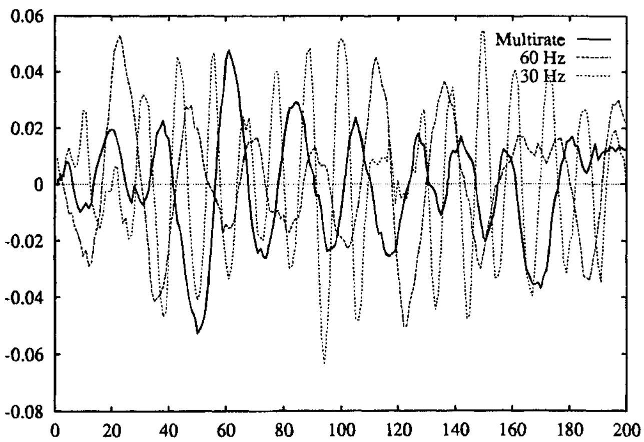

Fig. 2 Error state 7 vs sample number.

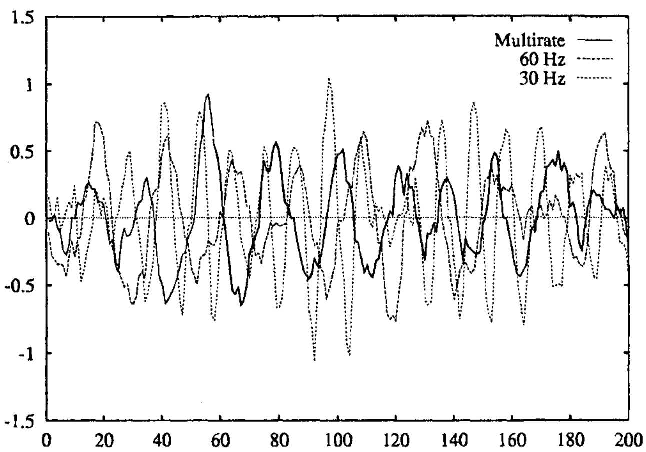

Fig. 3 Error state 8 vs sample number.

VI. Conclusion

This paper has considered the reduced-order estimation problem for multirate systems. An equivalent multirate discrete-time representation was obtained for the given continuous-time system. Optimality conditions were derived for the problem of optimal reduced-order, multirate estimation. Furthermore, a novel homotopy continuation algorithm was developed to obtain numerical solutions to the full-order design equations. Future work will use these results to develop more sophisticated numerical algorithms for reduced-order, multirate estimator design.

Appendix A: Proof of Lemma 1

It follows from Eq. (18) that

$$\text{vec}\,\tilde{Q}(k+1) = [\tilde{A}(k) \otimes \tilde{A}(k)]\text{vec}\,\tilde{Q}(k) + \text{vec}\,\tilde{V}(k) \tag{A1}$$

where $\otimes$ denotes Kronecker product and vec is the column stacking operator.[29] Next, define the notation $q(k) \triangleq \text{vec}\,\tilde{Q}(k)$, $\mathcal{A}(k) \triangleq \tilde{A}(k) \otimes \tilde{A}(k)$, and $v(k) \triangleq \text{vec}\,\tilde{V}(k)$, so that

$$q(k+1) = \mathcal{A}(k)q(k) + v(k) \tag{A2}$$

It now follows with $k = \alpha + \beta N$ that

$$q(k+\beta N) = \Phi(\alpha+\beta N, 1)q(1) + \sum_{i=1}^{\alpha+\beta N-1} \Phi(\alpha+\beta N, i+1)v(i) \tag{A3}$$

where

$$\Phi(\alpha+\beta N, i+1) \triangleq \mathcal{A}(\alpha+\beta N-1)\mathcal{A}(\alpha+\beta N-2)\cdots\mathcal{A}(i+1)$$
$$\times\ \alpha+\beta N > i+1, \qquad \Phi(\alpha+\beta N, \alpha+\beta N) = I_{n^2} \tag{A4}$$

Next, note that

$$\sum_{i=1}^{\alpha+\beta N-1} \Phi(\alpha+\beta N, i+1)v(i) = \sum_{i=1}^{N+(\alpha-1)} \Phi(\alpha+\beta N, i+1)v(i) + \sum_{i=N+\alpha}^{2N+(\alpha-1)} \Phi(\alpha+\beta N, i+1)v(i) + \sum_{i=2N+\alpha}^{3N+(\alpha-1)} \Phi(\alpha+\beta N, i+1)v(i) + \cdots + \sum_{i=(\beta-1)N+\alpha}^{\beta N+(\alpha-1)} \Phi(\alpha+\beta N, i+1)v(i) \tag{A5}$$

Using the identities $\Phi(\alpha+\beta N, 1) = \Phi^{\beta}(\alpha+N, 1)$ and $\Phi(\alpha+\beta N, \alpha+\gamma N) = \Phi^{\beta-\gamma}(\alpha+N, \alpha)$, it now follows that Eq. (A5) is equivalent to

$$\sum_{i=1}^{\alpha+\beta N-1} \Phi(\alpha+\beta N, i+1)v(i) = \Phi^{\beta-1}(\alpha+N, \alpha) \sum_{i=1}^{N+(\alpha-1)} \Phi(\alpha+N, i+1)v(i) + \Phi^{\beta-2}(\alpha+N, \alpha) \sum_{i=\alpha}^{N+(\alpha-1)} \Phi(\alpha+N, i+1)v(i) + \Phi^{\beta-3}(\alpha+N, \alpha) \sum_{i=\alpha}^{N+(\alpha-1)} \Phi(\alpha+N, i+1)v(i) + \cdots + \sum_{i=\alpha}^{N+(\alpha-1)} \Phi(\alpha+N, i+1)v(i) \tag{A6}$$

which implies that

$$\sum_{i=1}^{\alpha+\beta N-1} \Phi(\alpha+\beta N, i+1)v(i) = \Phi_{\alpha}^{\beta-1} \sum_{i=1}^{\alpha-1} \Phi(\alpha+N, i+1)v(i) + [I + \Phi_{\alpha} + \Phi_{\alpha}^{2} + \cdots + \Phi_{\alpha}^{\beta-1}] \times \sum_{i=\alpha}^{N+(\alpha-1)} \Phi(\alpha+N, i+1)v(i) \tag{A7}$$

where $\Phi_{\alpha} \triangleq \Phi(\alpha+N, \alpha)$. Since $\rho(\Phi_{\alpha}) < 1$ it follows from Eqs. (A3) and (A7) that

$$\lim_{k\to\infty} q(\alpha+\beta N) = (I-\Phi_{\alpha})^{-1} \sum_{i=\alpha}^{N+(\alpha-1)} \Phi(\alpha+N, i+1)v(i) \tag{A8}$$

which shows that the second moment converges to a steady-state periodic trajectory.

To prove Eq. (28), rewrite Eq. (28) as

$$J_e[A_e(\cdot), B_e(\cdot), C_e(\cdot)] = \delta_{\infty} + \lim_{K\to\infty} \frac{1}{K} \operatorname{tr} \sum_{\alpha=1}^{K} [\tilde{Q}(\alpha)\tilde{R}(\alpha)] \tag{A9}$$

Because of the periodicity of the closed-loop second-moment matrix $\tilde{Q}(\cdot)$ we obtain

$$J_e[A_e(\cdot), B_e(\cdot), C_e(\cdot)] = \delta + \frac{1}{N} \operatorname{tr} \sum_{\alpha=1}^{N} [\tilde{Q}(\alpha)\tilde{R}(\alpha)] \tag{A10}$$

□

Appendix B: Proof of Theorem 2

To optimize Eq. (28) subject to constraint Eq. (26) over the open set $\mathfrak{S}$ form the Lagrangian

$$\mathscr{L}[A_e(\alpha), B_e(\alpha), C_e(\alpha), \tilde{Q}(\alpha), \tilde{P}(\alpha+1), \lambda] \triangleq \operatorname{tr} \sum_{\alpha=1}^{N} \left\{ \lambda \frac{1}{N} [\tilde{Q}(\alpha)\tilde{R}(\alpha)] + [(\tilde{A}(\alpha)\tilde{Q}(\alpha)\tilde{A}^{T}(\alpha) + \tilde{V}(\alpha) - \tilde{Q}(\alpha+1))\tilde{P}(\alpha+1)] \right\} \tag{B1}$$

where the Lagrange multipliers $\lambda \geq 0$ and $\tilde{P}(\alpha+1) \in \mathscr{R}^{(n+n_e)\times(n+n_e)}$, $\alpha = 1, \ldots, N$ are not all zero. Thus, we obtain

$$\frac{\partial \mathscr{L}}{\partial \tilde{Q}(\alpha)} = \tilde{A}^{T}(\alpha)\tilde{P}(\alpha+1)\tilde{A}(\alpha) + \lambda \frac{1}{N}\tilde{R}(\alpha) - \tilde{P}(\alpha), \quad \alpha = 1, \ldots, N \tag{B2}$$

Setting $[\partial\mathscr{L}/\partial\tilde{Q}(\alpha)] = 0$ yields

$$\tilde{P}(\alpha) = \tilde{A}^{T}(\alpha)\tilde{P}(\alpha+1)\tilde{A}(\alpha) + \lambda(1/N)\tilde{R}(\alpha), \quad \alpha = 1, \ldots, N \tag{B3}$$

Next, propagating Eq. (B3) from α to $\alpha + N$ yields

$$\begin{aligned}\tilde{P}(\alpha) &= \tilde{A}^{T}(\alpha)\cdots\tilde{A}^{T}(\alpha+N-1) \\ &\quad \times \tilde{P}(\alpha)\tilde{A}(\alpha+N-1)\cdots\tilde{A}(\alpha) \\ &\quad + \lambda(1/N)[\tilde{A}^{T}(\alpha)\cdots\tilde{A}^{T}(\alpha+N-2) \\ &\quad \times \tilde{R}(\alpha+N-1)\tilde{A}(\alpha+N-2)\cdots\tilde{A}(\alpha) \\ &\quad + \tilde{A}^{T}(\alpha)\cdots\tilde{A}^{T}(\alpha+N-3)\tilde{R}(\alpha+N-2) \\ &\quad \times \tilde{A}(\alpha+N-3)\cdots\tilde{A}(\alpha) + \cdots + \tilde{R}(\alpha)]\end{aligned} \tag{B4}$$

Note that since $\tilde{A}(\alpha+N-1)\cdots\tilde{A}(\alpha)$ is assumed to be stable, $\lambda = 0$ implies $\tilde{P}(\alpha) = 0$, $\alpha = 1, \ldots, N$. Hence, it can be assumed without loss of generality that $\lambda = 1$. Furthermore, $\tilde{P}(\alpha)$, $\alpha = 1, \ldots, N$ is non-negative definite.

Now partition the $(n+n_e) \times (n+n_e)$ matrices $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha)$ into $n \times n$, $n \times n_e$, and $n_e \times n_e$ subblocks as

$$\tilde{Q}(\alpha) = \begin{bmatrix} Q_1(\alpha) & Q_{12}(\alpha) \\ Q_{12}^{T}(\alpha) & Q_2(\alpha) \end{bmatrix}, \qquad \tilde{P}(\alpha) = \begin{bmatrix} P_1(\alpha) & P_{12}(\alpha) \\ P_{12}^{T}(\alpha) & P_2(\alpha) \end{bmatrix}$$

Thus, with $\lambda = 1$, the stationary conditions are given by

$$\frac{\partial \mathscr{L}}{\partial \tilde{P}(\alpha+1)} = \tilde{A}(\alpha)\tilde{Q}(\alpha)\tilde{A}^{T}(\alpha) + \tilde{V}(\alpha) - \tilde{Q}(\alpha+1) = 0 \tag{B5}$$

$$\frac{\partial \mathscr{L}}{\partial A_e(\alpha)} = P_{12}^{T}(\alpha+1)A(\alpha)Q_{12}(\alpha) + P_2(\alpha+1)B_e(\alpha)C(\alpha)Q_{12}(\alpha) + P_2(\alpha+1)A_e(\alpha)Q_2(\alpha) = 0 \tag{B6}$$

$$\frac{\partial \mathcal{L}}{\partial B_e(\alpha)} = P_{12}^T(\alpha+1)A(\alpha)Q_1(\alpha)C^T(\alpha) + P_2(\alpha+1)B_e(\alpha)C(\alpha)Q_1(\alpha)C^T(\alpha) + P_2(\alpha+1)A_e(\alpha)Q_{12}^T(\alpha)C^T(\alpha) + P_{12}^T(\alpha)V_{12}(\alpha) + P_2(\alpha+1)B_e(\alpha)V_2(\alpha) = 0 \quad \text{(B7)}$$

$$\frac{\partial \mathcal{L}}{\partial C_e(\alpha)} = -R(\alpha)\hat{L}(\alpha)Q_{12}(\alpha) + R(\alpha)C_e(\alpha)Q_2(\alpha) = 0 \quad \text{(B8)}$$

for $\alpha = 1, \ldots, N$. Expanding Eqs. (26) and (B3) yields

$$0 = A(\alpha)Q_1(\alpha)A^T(\alpha) + V_1(\alpha) - Q_1(\alpha+1) \quad \text{(B9)}$$

$$0 = A(\alpha)Q_1(\alpha)C^T(\alpha)B_e^T(\alpha) + A(\alpha)Q_{12}(\alpha)A_e^T(\alpha) + V_{12}(\alpha)B_e^T(\alpha) - Q_{12}(\alpha+1) \quad \text{(B10)}$$

$$0 = B_e(\alpha)C(\alpha)Q_1(\alpha)C^T(\alpha)B_e^T(\alpha) + A_e(\alpha)Q_{12}^T(\alpha)C^T(\alpha)B_e^T(\alpha) + B_e(\alpha)C(\alpha)Q_{12}(\alpha)A_e^T(\alpha) + A_e(\alpha)Q_2(\alpha)A_e^T(\alpha) + B_e(\alpha)V_2(\alpha)B_e^T(\alpha) - Q_2(\alpha+1) \quad \text{(B11)}$$

$$0 = A^T(\alpha)P_1(\alpha+1)A(\alpha) + C^T(\alpha)B_e^T(\alpha)P_{12}^T(\alpha+1)A(\alpha) + A^T(\alpha)P_{12}(\alpha+1)B_e(\alpha)C(\alpha) + C^T(\alpha)B_e^T(\alpha)P_2(\alpha+1)B_e(\alpha)C(\alpha) + (1/N)R_1(\alpha) - P_1(\alpha) \quad \text{(B12)}$$

$$0 = A^T(\alpha)P_{12}(\alpha+1)A_e(\alpha) + C^T(\alpha)B_e^T(\alpha)P_2(\alpha+1)A_e(\alpha) + (1/N)R_{12}(\alpha) - P_{12}(\alpha) \quad \text{(B13)}$$

$$0 = A_e^T(\alpha)P_2(\alpha+1)A_e(\alpha) + (1/N)R_2(\alpha) - P_2(\alpha) \quad \text{(B14)}$$

Lemma 3. $Q_2(\alpha)$ and $P_2(\alpha)$ are positive definite for $\alpha = 1, \ldots, N$.

Proof: By a minor extension of the results from Ref. 30, Eq. (B11) can be written as

$$Q_2(\alpha+1) = [A_e(\alpha) + B_e(\alpha)C(\alpha)Q_{12}(\alpha)Q_2^+(\alpha)]Q_2(\alpha) \times [A_e(\alpha) + B_e(\alpha)C(\alpha)Q_{12}(\alpha)Q_2^+(\alpha)]^T + B_e(\alpha)V_{2a}(\alpha)B_e^T(\alpha), \quad \alpha = 1, \ldots, N \quad \text{(B15)}$$

where $Q_2^+(\alpha)$ is the Moore-Penrose or Drazin generalized inverse of $Q_2(\alpha)$. Next, propagating Eq. (B15) from α to $\alpha + N$ yields

$$Q_2(\alpha) = A_{es}(\alpha+N-1)\cdots A_{es}(\alpha)Q_2(\alpha)A_{es}^T(\alpha)\cdots A_{es}^T(\alpha+N-1) + A_{es}(\alpha+N-1)\cdots A_{es}(\alpha+1)B_e(\alpha)V_{2a}(\alpha)B_e^T(\alpha)A_{es}^T(\alpha+1)\cdots A_{es}^T(\alpha+N-1) + A_{es}(\alpha+N-1)\cdots A_{es}(\alpha+2)B_e(\alpha+1)V_{2a}(\alpha+1)B_e^T(\alpha+1)A_{es}^T(\alpha+2)\cdots A_{es}^T(\alpha+N-1) + \cdots + B_e(\alpha+N-1)V_{2a}(\alpha+N-1) \times B_e^T(\alpha+N-1) \quad \text{(B16)}$$

where $A_{es}(\cdot) \triangleq A_e(\cdot) + B_e(\cdot)C(\cdot)Q_{12}(\cdot)Q_2^+(\cdot)$. Next, note that the controllability of $[\Phi_{esp}(\alpha), B_{es}(\alpha)]$ implies that $[\Phi_{esp}(\alpha), B_{es}(\alpha)V_{2s}^{1/2}(\alpha)]$ is also controllable, where

$$\Phi_{esp}(\alpha) \triangleq A_{es}(\alpha+N-1)A_{es}(\alpha+N-2) \times A_{es}(\alpha+N-3)\cdots A_{es}(\alpha)$$

$$B_{es}(\alpha) \triangleq [A_{es}(\alpha+N-1)\cdots A_{es}(\alpha+1)B_e(\alpha), A_{es}(\alpha+N-1)\cdots A_{es}(\alpha+2) \cdot B_e(\alpha+1), \ldots, B_e(\alpha+N-1)]$$

$$V_{2s}(\alpha) \triangleq \text{block-diagonal } [V_{2a}(\alpha), V_{2a}(\alpha+1), \ldots, V_{2a}(\alpha+N-1)]$$

Next, using the given notation, Eq. (B16) becomes

$$Q_2(\alpha) = \Phi_{esp}(\alpha)Q_2(\alpha)\Phi_{esp}^T(\alpha) + B_{es}(\alpha)V_{2s}(\alpha)B_{es}^T(\alpha) \quad \text{(B17)}$$

Now, since $\Phi_{esp}(\cdot)$ is stable and $B_{es}(\cdot)V_{2s}(\cdot)B_{es}^T(\cdot)$ is non-negative definite, Lemma 12.2′ (p. 282, Ref. 31) implies that $Q_2(\cdot)$ is positive definite. Using similar arguments we can show that $P_2(\cdot)$ is positive definite. □

Since $Q_2(\alpha)$, $P_2(\alpha)$, $V_2(\alpha)$, $R_2(\alpha)$, $V_{2a}(\alpha)$, and $R_{2a}(\alpha)$, for $\alpha = 1, \ldots, N$, are invertible Eqs. (B6–B8) can be written as

$$A_e(\alpha) = -P_2^{-1}(\alpha+1)P_{12}^T(\alpha+1)A(\alpha)Q_{12}(\alpha)Q_2^{-1}(\alpha) - B_e(\alpha)C(\alpha)Q_{12}(\alpha)Q_2^{-1}(\alpha) \quad \text{(B18)}$$

$$B_e(\alpha) = -P_2^{-1}(\alpha+1)P_{12}^T(\alpha+1)Q_a(\alpha)V_{2a}^{-1}(\alpha) \quad \text{(B19)}$$

$$C_e(\alpha) = \hat{L}(\alpha)Q_{12}(\alpha)Q_2^{-1}(\alpha) \quad \text{(B20)}$$

for $\alpha = 1, \ldots, N$. Now, for $\alpha = 1, \ldots, N$, define $n_e \times n$, $n_e \times n_e$, and $n_e \times n$ matrices

$$G(\alpha) \triangleq Q_2^{-1}(\alpha)Q_{12}^T(\alpha), \qquad M(\alpha) \triangleq Q_2(\alpha)P_2(\alpha)$$

$$\Gamma(\alpha) \triangleq -P_2^{-1}(\alpha)P_{12}^T(\alpha)$$

and the $n \times n$ matrices

$$Q(\alpha) \triangleq Q_1(\alpha) - Q_{12}(\alpha)Q_2^{-1}(\alpha)Q_{12}^T(\alpha)$$

$$P(\alpha) \triangleq P_1(\alpha) - P_{12}(\alpha)P_2^{-1}(\alpha)P_{12}^T(\alpha)$$

$$\hat{Q}(\alpha) \triangleq Q_{12}(\alpha)Q_2^{-1}(\alpha)Q_{12}^T(\alpha)$$

$$\hat{P}(\alpha) \triangleq P_{12}(\alpha)P_2^{-1}(\alpha)P_{12}^T(\alpha)$$

Next, for $\alpha = 1, \ldots, N$, computing $\Gamma(\alpha+1) \cdot$ [Eq. (B10)] − [Eq. (B11)] = 0 and $G(\alpha) \cdot$ [Eq. (B13)] + [Eq. (B14)] = 0 yields

$$-P_2^{-1}(\alpha+1)P_{12}^T(\alpha+1)Q_{12}(\alpha+1)Q_2^{-1}(\alpha+1) = I_{n_e} \quad \text{(B21)}$$

$$-P_2^{-1}(\alpha)P_{12}^T(\alpha)Q_{12}(\alpha)Q_2^{-1}(\alpha) = I_{n_e} \quad \text{(B22)}$$

Note, $Q(\alpha)$, $P(\alpha)$, $\hat{P}(\alpha)$, and $\hat{Q}(\alpha)$, for $\alpha = 1, \ldots, N$, are non-negative definite. Next, note that with the preceding definitions (B21) and (B22) are equivalent to Eq. (33b), and Eq. (33a) holds. Hence, $\tau(\alpha) = G^T(\alpha)\Gamma(\alpha)$ is idempotent, i.e., $\tau^2(\alpha) = \tau(\alpha)$, for $\alpha = 1, \ldots, N$. Sylvester's inequality yields Eq. (42). Note also that

$$\hat{Q}(\alpha) = \tau(\alpha)\hat{Q}(\alpha), \qquad \hat{P}(\alpha) = \hat{P}(\alpha)\tau(\alpha)$$

The components of $\tilde{Q}(\alpha)$ and $\tilde{P}(\alpha)$ can now be written in terms of $Q(\alpha)$, $P(\alpha)$, $\hat{Q}(\alpha)$, $\hat{P}(\alpha)$, $G(\alpha)$, and $\Gamma(\alpha)$ as

$$Q_1(\alpha) = Q(\alpha) + \hat{Q}(\alpha), \qquad P_1(\alpha) = P(\alpha) + \hat{P}(\alpha)$$

$$Q_{12}(\alpha) = \hat{Q}(\alpha)\Gamma^T(\alpha), \qquad P_{12}(\alpha) = -\hat{P}(\alpha)G^T(\alpha)$$

$$Q_2(\alpha) = \Gamma(\alpha)\hat{Q}(\alpha)\Gamma^T(\alpha), \qquad P_2(\alpha) = G(\alpha)\hat{P}(\alpha)G^T(\alpha)$$

The expressions (36–38) follow from Eqs. (B6–B8) by using the preceding identities. Substituting these expressions for $A_e(\alpha)$, $B_e(\alpha)$, and $C_e(\alpha)$ in Eqs. (B9–B14) it follows that Eq. (B11) = $\Gamma(\alpha + 1) \cdot$ [Eq. (B10)] and Eq. (B14) = $-G(\alpha) \cdot$ [Eq. (B13)]. Hence, it follows that Eqs. (B11) and (B14) are superfluous and can be omitted. Thus, Eqs. (B9–B14) reduce to

$$0 = A(\alpha)Q(\alpha)A^T(\alpha) + A(\alpha)\hat{Q}(\alpha)A^T(\alpha) + V_1(\alpha) - Q(\alpha + 1) - \hat{Q}(\alpha + 1) \tag{B23}$$

$$0 = [A(\alpha)\hat{Q}(\alpha)A^T(\alpha) + Q_a(\alpha)V_{2a}^{-1}(\alpha)Q_a^T(\alpha) - \hat{Q}(\alpha + 1)]\Gamma^T(\alpha + 1) \tag{B24}$$

$$0 = [\{A(\alpha) - Q_a(\alpha)V_{2a}^{-1}(\alpha)C(\alpha)\}^T\hat{P}(\alpha + 1)\{A(\alpha) - Q_a(\alpha)V_{2a}^{-1}(\alpha)C(\alpha)\} + (1/N)\hat{L}^T(\alpha)R\hat{L}(\alpha) - \hat{P}(\alpha)]G^T(\alpha) \tag{B25}$$

Next, computing Eq. (B23) + $G^T(\alpha + 1)\Gamma(\alpha + 1) \cdot$ [Eq. (B24)]$G(\alpha + 1)$ − [Eq. (B24)]$G(\alpha + 1)$ − {[Eq. (B24)]$G(\alpha + 1)\}^T = 0$, yields Eq. (39). Finally Eqs. (40) and (41) are obtained by computing $G^T(\alpha + 1)\Gamma(\alpha + 1) \cdot$ [Eq. (B24)]$G(\alpha + 1) = 0$ and $\Gamma^T(\alpha)G(\alpha) \cdot$ [Eq. (B25)]$\Gamma(\alpha) = 0$, respectively. □

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